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Problémy přiřazení pólů v nečtvercových lineárních systémech

Autor: Ing. Tetiana Korotka

Školitel: Ing. Petr Zagalak, Csc.

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Abstrakt

Tato dizertační práce se zabývá teorií nečtvercových systémů, což je nejobecnější případ lineárních implicitních systémů. Konkrétněji, práce se zabývá vlivem stavové zpětné vazby na změnu chování výše zmíněných systémů. Hlavní pozornost je pak věnována problémům přiřazení pólů či struktury pólů. Tyto problémy patří k těm nejdůležitějším v teorii lineárního řízení, protože se zaměřují na změnu chování zpětnovazebního obvodu volbou vhodné stavové zpětné vazby. Výsledkem je serie několika vět, které přispívají k řešení zmíněných problémů. Za hlavní výsledek práce lze považovat Theorem 5, kde jsou uvedeny nutné a postačující podmínky pro přiřazeí pólů v tzv. sloupcově regularizovatelných systémech.

Teorie čtvercových (regularizovatelných) systémů je v současné době již dobře rozvinuta, avšak rozšíření jejích výsledků na nečtvercové systémy není zdaleka triviální. Na vině jsou některé speciální vlastnosti nečtvercových systémů, jako je např. nejedoznačnost stavových trajektorií či dodatečné podmínky na řídící vstup, které vedou k existování stavové trajektorie. Vzhledem k těmto vlastnostem bylo proto nutné přezkoumat pojem řiditelnosti a navrhnout její novou definici, která se přechází v původní definici v případě čtvercových systémů.

Teorie nečtvercových systémů je užitečná při studiu velké škály systémů, např. ekonomických, biologických nebo elektronických, takže pole aplikovatelnosti dosažených výsledků může být velmi široké.

Kličová slova: lineární systémy, zpetná vazba, přiřazení pólu

Abstract

The thesis is devoted to the theory of non-square systems - the most general case of linear implicit systems which is described by the non-square matrices. In particular, the effect of a proportional state feedback upon the behavior of such systems is investigated. The main attention in the work is given to the problems of pole and pole structure assignment by feedback. These problems belong to the most important ones in control theory. They aim at the assignment of the closed-loop behavior of the system by choosing the appropriate state feedback gain. As a result, a series of theorems contributing to the problems of pole and pole structure assignment by state feedback for the non-square systems were established. The main contribution is given in Theorem 5 which states the necessary and sufficient conditions for the pole assignment problem in the so-called column regularizable systems.

The class of square systems has been widely studied. But the results as well as some concepts known for them can not be, in general, extended for the non-square systems in an appropriate way. In fact, the non-square systems possess some special features like the non-uniqueness of the state trajectory or the presence of additional constraints upon the control input to ensure the existence of the trajectory. Taking that into account, the concepts of control that are closely related to the problem of pole assignment, the most important of which is the controllability, are analyzed. Then, in accordance with the task of pole assignment by a proportional feedback, analogues of that concepts are proposed (which reduce to the classical ones in the case of square systems).

The theory of non-square systems is very useful for describing and studying a large variety of systems like economic, circuit, biological etc. Hence, in our opinion, the application of the achieved results can be very broad.

Keywords: linear systems, state feedback, pole assignment

Contents

1 Introduction

In the thesis, the main subject of the study is a non-square linear system of the form

$$E\dot{x}(t) \;=\; Ax(t) + Bu(t), \quad t \geq 0, \tag{1}$$

where $x(t), u(t)$ is the state, the initial value of which is $x(0) = x_0$, and the control input, respectively, and $E, A \in \mathbb{R}^{q\times n}$, $B \in \mathbb{R}^{q\times m}$. In particular, the effect of state feedback upon the system is investigated.

The system (1) will frequently be referred to as the triple (E, A, B). It is called *non-square* or *rectangular* since q, in general, does not equal n. Hence, the matrices E and A are non-square. The special cases of such systems are recalled below.

- (E, A, B) is called *square* if $q = n$;
- (E, A, B) is called *explicit* if $E \sim I_n$ otherwise it is *implicit*;
- (E, A, B) is called *regular* if the pencil $sE - A$ is regular (that is when $\det[sE - A] \neq 0$);
- (E, A, B) is called *regularizable* (by a proportional state feedback) if there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that the pencil $sE - A - BF$ is regular.

The square systems (1) are assumed here to be regularizable - this guaranties the existence of its transfer function. Such systems have been widely studied. The explicit systems were investigated at the first stage, and the main concepts of control theory were introduced to describe their properties. When dealing with the regular, and then regularizable systems, some new effects that had not properly been described by the existing concepts were revealed. Hence the application of the results known for the explicit systems was not suitable for some tasks. Similarly, rectangular systems possess some special features. And of course, their study is of theoretical and practical interest (the rectangular systems comprise the square systems as the special case).

The theory of non-square systems provides a suitable mathematical tool for describing a large variety of systems like economic, production, biological, physical etc., see [2, 15, 24] and references therein. Robotics is another area where many models are in the form of non-square (generally nonlinear) systems and their linearization may lead to the systems of the form (1). The non-square systems frequently appear in modeling some networks like signal flow graphs, Petri nets,.... and can be applied to the fields like circuit systems, composite systems. For example, in network analysis, there are several effects, like hysteresis, that can be modeled using rectangular systems of differential equations [19]. For the review of some applications of the non-square systems see [11, 23, 28].

One of the basic problems of control lies in altering the behavior of the system (1) so that a desirable performance is achieved. This is realized by means of control inputs among of which the state feedback is very useful. The dynamical behavior of the system is strongly influenced by its pole structure. Therefore, the description of all the possible pole structures of the system that can be assigned by different feedback gains is of great importance in control. The problem of pole structure assignment, here by state feedback, treats with such a question. The closely related problem, called as the pole assignment problem, deals with the question of existence of a state feedback gain such that the prescribed poles will be assigned to the system. These problems constitute to the fundamental problems of control as they aim at shaping the desired closed-loop system response. Hence, the pole assignment techniques belong to the basic tools for the controller design. One can meet many modifications of these problems, see for example [16, 20, 22, 31] and the references therein. The case when just the finite pole structure is assigned to the system (1) resulting in the elimination of the so called impulsive behavior of the system is of great practical interest [18]. An assignment of the pure infinite eigenvalue structure to the closed-loop system is also important in control. As an example, the design of perfect observers can be considered [32].

Pole and pole structure assignment problems are widely studied by many authors in the case of square systems. For the explicit systems the problem of pole structure assignment has been completely solved, see [8, 13, 17, 29, 31, 36]. The necessary and sufficient conditions for the assignment of prescribed pole structure to the implicit and controllable square systems are also known [18, 37]. Recent results for the same problem were given for the regularizable systems [9, 22]. The problem of pole assignment and its special cases for the square systems is also well studied [4, 5, 9, 12, 22, 26, 27]. As far as concerns the rectangular systems the literature on this topic is not very extensive, [9, 10]. In particular, the authors solved there the problem of pole structure assignment for some special cases of (1).

Such a notion as controllability of the system plays an important role in the pole assignment problems. It describers the abilities for the shifting the poles to the prescribed locations by means of control input. Controllability for the case of square systems was deeply studied by different approaches like time-domain, frequency-domain etc. [6, 7, 19, 30, 31, 34, 35]. The case of rectangular systems is less studied. For such systems there are a few papers [3, 14] devoted to this notion. But since controllability is considered there to be adapted to different tasks the approaches to it are different. This calls for further study in this direction especially in relation with the problem of pole assignment by a proportional state feedback.

1.1 Problem formulation

Applying the (linear and proportional) state feedback

$$u(t) \;=\; Fx(t) + v(t), \tag{2}$$

where $v(t)$ is a new control input and $F \in \mathbb{R}^{m\times n}$ is a state feedback gain, yields the closed-loop system

$$E\dot{x}(t) \;=\; [A + BF]x(t) + Bv(t). \tag{3}$$

Under the Laplace transform of (3) (including (1) for $F = 0$) the system is written as

$$[sE - A - BF]X(s) \;=\; BU(s) + Ex_0, \tag{4}$$

with $X(s), U(s)$ denoting the Laplace transforms of $x(t), u(t)$, respectively. The rank of the pencil $sE - A - BF$ is denoted by r.

The **pole structure** of the system (E, A, B) is defined by the zero structure of the pencil $sE - A$ [22] . The *finite zero structure* of $sE - A$ is given by the invariant polynomials of $sE-A$, say $\psi_i(s) \triangleright \psi_{i+1}(s),\ i = 1, \ldots, r-1$. The *infinite zero structure* is defined [1, 33] by the infinite elementary divisors of $sE - A$ of the orders $\mu_i > 1$, $\mu_i - 1 = d_i$, where $d_i,\ i = 1, \ldots, k_d$, are called the infinite zero orders.

The **poles** of the system (E, A, B) are defined by the (finite and infinite) zeros of $sE - A$. The *finite poles* are described by the polynomial $\psi(s) := \prod_{i=1}^{r} \psi_i(s)$. The *pole at infinity* is described by its multiplicity $d := \sum_{i=1}^{k_d} d_i$.

The differences between the systems (1) and (3) are mainly given by the changes in the zero structure of $sE - A - BF$ when varying F. By choosing different state feedback gains F we alter the zero structure of $sE - A - BF$, and consequently the response of the system. This problem is called as the pole structure assignment and is formulated as follows [22].

Pole Structure Assignment Problem:

Given a system (1), monic polynomials $\psi_1(s) \triangleright \psi_2(s) \triangleright \ldots \triangleright \psi_r(s)$, and positive integers $d_1 \geq d_2 \geq \ldots \geq d_{k_d}$, find necessary and sufficient conditions for the existence of a matrix F in (2) such that the polynomials $\psi_i(s)$ and integers d_i will be the invariant polynomials and infinite zero orders of $sE - A - BF$, respectively.

A special case of the pole structure assignment problem is the problem of pole assignment (characteristic polynomial assignment in the case of explicit systems).

Pole Assignment Problem:

Given a system (1), monic polynomial $\psi(s)$, and positive integer d, find necessary and sufficient conditions for the existence of a matrix F in (2) such that the polynomial $\psi(s)$ and integer d will be the product of the invariant polynomials and sum of the infinite zero orders of $sE - A - BF$, respectively.

1.2 State of the art

The most general solutions of the problems of pole and pole structure assignment to the systems (1) by state feedback (2) till now are described below (for the notation of the indices see Feedback Canonical Form of (1) in Section 3).

- Necessary and sufficient conditions for the problem of pole assignment have been established for the regularizable systems (see for details Theorem 1 here), that is for the following case of (1)
  - $k_\eta = 0$ and $k_\epsilon = k_q$, see Theorem 1 in [22], Theorem 8 in [9].
- The conditions for the problem of pole structure assignment have been given for the systems (1) satisfying
  - $k_\eta = 0$ and $k_\epsilon = k_q$, see Theorem 5 in [22];
  - $k_\eta = 0$ and $k_\epsilon = k_q$, and $\text{card}\{\epsilon_i = 0\} = 0$, see Theorem 5 in [9];
  - $k_\eta = 0$ and $\text{card}\{\epsilon_i = 0\} = 0$, see Theorem 6 in [9].

  In the first case there given just necessary conditions which are sufficient in special cases (see Theorem 10 and Remark 1 here).

  In the last two cases the problem is completely solved.

Controllability has been well studied for the regularizable systems (1). Concerning the rectangular systems there a few papers [3, 14] devoted to this concept. One can notice that the approach to controllability for the systems when $q < n$ is different. This is caused by that this concept was considered there to be adapted to different tasks. Namely, controllability was considered in relation to the possibility of the existence of the prescribed state trajectory in [14] and the abilities for pole assignment by a proportional and derivative feedback in [3].

2 Goals and objectives of the dissertation

Specific goals of the thesis were set as follows.

- Extend the concept of controllability known for the square systems to the rectangular systems (1).

  This concept is very closely connected with the pole and pole structure assignment problems since it describes our possibilities for shifting the poles of the system (1).

- Give the characterization of controllability indices for the non-square systems.

  The notion of controllability indices is very useful when treating the problem of pole structure assignment to the systems (1).

- Contribute to the problem of pole structure assignment by state feedback (2) for the case of rectangular systems (1).

  This problem is of significant value in control and it has not been yet well studied in the case of non-square systems (1).

- Contribute to the problem of pole assignment by state feedback (2) for the rectangular systems (1).

  This problem is also of great theoretical as well as practical interest and has not been yet well studied for the general case of (E, A, B).

3 Main tools

The system (1) is assumed to be in the **Feedback Canonical Form** [21]. This is very useful to investigate the influence of state feedback upon the system. In particular, using a quadruple of the matrices P, Q, G, F, where P, Q, G are invertible matrices and F is an $m \times n$ matrix over $\mathbb{R}$, that acts as a transformation, as follows

$$(P, Q, G, F) \circ (E, A, B) = (PEQ, P[A + BF]Q, PBG) := (E_C,\ A_C,\ B_C),$$

each system (E, A, B) can be brought into the Feedback Canonical Form. The pencil $sE_C - A_C$ consists of some of the following pencils,

$$sE_C - A_C := \text{block diag}\left\{sE_j - A_j\right\},\ j \in \{\epsilon, \sigma, q, p, l, \eta\},$$

where $sE_j - A_j$ is again a block diagonal matrix pencil consisting of the blocks, non-increasingly ordered by size, of types (b_j), $j \in \{\epsilon, \sigma, q, p, l, \eta\}$,

$$(b_\epsilon)\ \overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}}^{\epsilon_i+1}\Big\}\epsilon_i \qquad (b_\sigma)\ \overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s \end{bmatrix}}^{\sigma_i}\Big\}\sigma_i$$

$$(b_q)\ \overbrace{\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}}^{q_i}\Big\}q_i+1 \qquad (b_p)\ \overbrace{\begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}}^{p_i+1}\Big\}p_i+1$$

$$(b_l)\ \overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -a_{i0} & -a_{i1} & \cdots & s-a_{il_i} \end{bmatrix}}^{l_i}\Big\}l_i \qquad (b_r)\ \overbrace{\begin{bmatrix} s & & \\ -1 & \ddots & \\ & \ddots & s \\ & & -1 \end{bmatrix}}^{\eta_i}\Big\}\eta_i+1,$$

The values describing these blocks are called:

- the nonproper controllability indices, $\epsilon_1 \geq \ldots \geq \epsilon_{k_\epsilon} \geq 0$;
- the proper controllability indices, $\sigma_1 \geq \ldots \geq \sigma_{k_\sigma} > 0$;
- the almost proper controllability indices, $q_1 \geq \ldots \geq q_{k_q} \geq 0$;
- the almost nonproper controllability indices, $p_1 \geq \ldots \geq p_{k_p} \geq 0$;
- the fixed invariant polynomials of $[sE_C - A_C \ \ -B_C]$ given by the polynomials $\alpha_i(s) = s^{l_i} + a_{il_i}s^{l_i-1} + \cdots + a_{i1}s + a_{i0}$, $l_i > 0$, which satisfy $\alpha_1(s) \triangleright \alpha_2(s) \triangleright \cdots \triangleright \alpha_{k_l}(s)$;
- the row minimal indices of $[sE_C - A_C \ \ -B_C]$, $\eta_1 \geq \ldots \geq \eta_{k_\eta} \geq 0$.

The matrix B_C is of the form

$$B_C := \begin{bmatrix} 0 & 0 \\ B_\sigma & 0 \\ 0 & B_q \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } \begin{cases} B_\sigma := \operatorname{block\,diag}\left\{[0\ldots0\ 1]^T \in \mathbb{R}^{\sigma_i}\right\}_{i=1}^{k_\sigma}, \\ B_q := \operatorname{block\,diag}\left\{[0\ldots0\ 1]^T \in \mathbb{R}^{q_i+1}\right\}_{i=1}^{k_q}. \end{cases}$$

The concept of **Normal External Description** (NED) [25] appears to be very useful in control theory, especially when approaching the problems involving state feedback. Polynomial matrices $N(s)$, $D(s)$ are said to form a NED of the system (E, A, B) if they satisfy the following conditions:

- $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ forms a minimal polynomial basis for $\text{Ker}[sE - A, \ -B]$;
- $N(s)$ forms a minimal polynomial basis for $\text{Ker}\Pi[sE - A]$, where Π is a maximal left annihilator of B,

The NED of the system (1) reflects just those parts of $[sE - A, -B]$ that are given by the $\epsilon-$ and $\sigma-$blocks. Let $\bar{B}$ be such that $[B\ \bar{B}]$ is of full column rank and $\text{rank}\,[sE - A, \ -[B\ \bar{B}]] = q$, $\forall s \in \mathbb{C} \cup \infty$. The system $(E, A, [B\ \bar{B}])$ obtained from (E, A, B) by this trick is called the *extended system* [22]. Let its NED be denoted as $\begin{bmatrix} N_E(s) \\ D_E(s) \end{bmatrix}$. The relationship between the closed-loop system and its NED is the following [22]:

- The non-unit invariant polynomials of both $sE - A - BF$ and $D_{EF}(s)$ coincide for any F, where $D_{EF}(s) := D_E(s) - \begin{bmatrix} F \\ 0 \end{bmatrix} N_E(s)$.

To handle the finite and infinite poles of (1) in a unified way, the **conformal mapping** [16] $s = \frac{1+aw}{w}$, where $a \in \mathbb{R}$, and is not a pole of (E, A, B), is used. Then, the point $s = \infty$ is moved to $w = 0$, while all the finite points except $s = a$ are kept in finite positions. By applying the conformal mapping, the pole structure assignment problem reduces to the description of just the finite zero structure of $\left[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\right]$, the w-analogue of $sE - A - BF$. In particular, this pencil posses the following structure

$$\psi_i(w) := w^{d_i + \deg \psi_i(s)} \psi_i \left(\frac{1 + aw}{w} \right) := w^{d_i} \tilde{\psi}_i \left(w \right),$$

where d_i and $\tilde{\psi}_i\left(w\right)$ are the infinite zero orders and w-analogues of invariant polynomials $\psi_i(s)$ of $sE - A - BF$, respectively.

Using the concept of NED, the zero structure of $\tilde{D}_{EF}(w)$, the w-analogue of $D_{EF}(s)$, is investigated. The matrix $\tilde{D}_{EF}(w)$ is, in general, of the form

$$\tilde{D}_{EF}(w)=\left[\begin{array}{cccccc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} & \tilde{D}_{1q} & \tilde{D}_{1p} & \tilde{D}_{1l} & \tilde{D}_{1\eta} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} & \tilde{S}_q+\tilde{D}_{2q} & \tilde{D}_{2p} & \tilde{D}_{2l} & \tilde{D}_{2\eta} \\ \hline 0 & 0 & Z_q & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{S}_\eta \end{array}\right] \tag{5}$$

where $\tilde{S}_\sigma := \mathrm{diag}\,\{(1+aw)^{\sigma_i}\}_{i=1}^{k_\sigma}$, $\quad \tilde{S}_q := \mathrm{diag}\,\{(1+aw)^{q_i}\}_{i=1}^{k_q}$,

$$Z_q := \mathrm{diag}\,\{-w^{q_i}\}_{i=1}^{k_q}, \qquad Z_p := \mathrm{diag}\,\{-w^{p_i}\}_{i=1}^{k_p},$$

$$\tilde{S}_\alpha := \mathrm{diag}\,\{\tilde{\alpha}_i(w)\}_{i=1}^{k_l} \qquad \tilde{S}_\eta := \mathrm{blockdiag}\left\{\left[\begin{array}{c}(1+aw)^{\eta_i} \\ -w^{\eta_i}\end{array}\right]\right\}_{i=1}^{k_\eta},$$

and $\tilde{D}_{ij}$ are polynomial matrices satisfying the conditions:

$$\deg_{ci}\left[\begin{array}{c}\tilde{D}_{1j} \\ \tilde{D}_{2j}\end{array}\right] \le j_i, \quad j \in \{\epsilon,\sigma,q,p,l,\eta\}, \; i=1,2,\ldots. \tag{6}$$

Particularly, the problem of pole structure assignment to the system (1) by a state feedback (2) is treated in the following way.

Approach to the Pole Structure Assignment Problem:

Given a system (1), monic polynomials $\psi_1(s) \triangleright \psi_2(s) \triangleright \ldots \triangleright \psi_r(s)$*, and positive integers* $d_1 \ge d_2 \ge \ldots \ge d_{k_d}$*, find necessary and sufficient conditions under which there exist matrices* $\tilde{D}_{ij}$ *satisfying (6) such that, using the* w*-notation, the polynomials* $\tilde{\psi}_1(w)w^{d_1} \triangleright \tilde{\psi}_2(w)w^{d_2} \triangleright \ldots \triangleright \tilde{\psi}_r(w)w^{d_r}$ *(*$d_i := 0$ *for* $i > k_d$*), will be the invariant polynomials of* $\tilde{D}_{EF}(w)$*, defined in (5).*

4 The problem of pole assignment

The recent result of the pole assignment problem for the systems (1) is presented below.

Theorem 1. *[22] Given a regularizable system (1) ($k_\epsilon = k_q$ and $k_\eta = 0$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$, then there exists a matrix F in (2) such that $\det[sE - A - BF] = \psi(s)$ and the sum of the infinite zero orders of $sE - A - BF$ equals d if and only if the conditions (7)-(9) (and (10) if $k_\epsilon = 0$) are satisfied:*

$$\deg \psi(s) + d = \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i \quad (7)$$

$$\psi(s) \triangleright \alpha_1(s)\alpha_2(s)...\alpha_{k_l}(s) \quad (8)$$

$$d \geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i \quad (9)$$

$$\deg \psi(s) = \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_l} l_i \quad (10)$$

Pole assignment to the non-square systems

When dealing with the non-square systems (1), a natural question arising here is under what conditions there exists a state feedback (2) yielding a full rank pencil $sE-A-BF$. This seems to be a reasonable property of a linear system (1) since it garanties the existence of (possible non-unique) transfer function. Then, by analogue with square systems, the concept of weak regularizability of (1) is introduced [4]. In particular,

- (E, A, B) is called a *weakly (column or row) regularizable* system if the pencil $sE - A - BF$ is of full column or row rank for some F in (2).

Theorem 2 gives an answer to the posed question:

(a) (E, A, B) is row regularizable iff $k_\epsilon \geq k_q$ and $k_\eta = 0$;

(b) (E, A, B) is column regularizable iff $k_q \geq k_\epsilon$.

The assumption of weak regularizability of the rectangular systems (1) plays a central role in the problem of pole assignment. In particular, the weak regularizability guaranties that (at least) one of the minors of largest possible order $\min\{q, n\}$ of $sE - A - BF$ is not zero. Let the minors of $P(s) \in \mathbb{R}^{q\times n}[s]$ of order $\min\{q,n\}$ be called *dominant* and be denoted as $\mathrm{dm}\,P(s)$. Then, the greatest common divisor of all dominant minors of $sE-A-BF$ (hereafter denoted by $\mathrm{gcddm}[sE-A-BF]$) plays a similar role as the determinant of the regular pencils. This means that the finite poles of the system $(E, A+BF, B)$ are given by the zeros of $\mathrm{gcddm}[sE-A-BF]$. Applying the conformal mapping and moving the point $s=\infty$ to $w=0$, and using the concept of the NED, it follows that the poles of the system are given by the zeros of $\mathrm{gcddm}\,\tilde{D}_{EF}(w)$.

Pole Assignment Problem in Weakly Regularizable Systems:

Given a weakly regularizable system (1), a monic polynomial $\psi(s)$, and integer $d>0$, find necessary and sufficient conditions under which there exist matrices $\tilde{D}_{ij}$ satisfying (6) such that, using the w-notation, $\tilde{\psi}(w)w^d$ will be a $\mathrm{gcddm}\,\tilde{D}_{EF}(w)$ defined in (5).

Then, using the methods of linear algebra, mainly the Laplace expansion, a series of theorems contributing to the pole assignment problem by state feedback in the case of the non-square systems was established.

- In the case of the **row regularizable systems** the main result is stated in **Theorem 3** and describes the necessary conditions of solvability to the problem of pole assignment.

Theorem 3. [2] *Let a row regularizable system (1) ($k_\epsilon \geq k_q$ and $k_\eta = 0$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$ be given. If there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that, using the w-notation, a $\tilde{\psi}(w)w^d = \mathrm{gcddm}\big[w\tilde{E}-\tilde{A}-\tilde{B}(w)F\big]$, then the conditions (11)- (13) (and (14) if $k_q=0$) are satisfied:*

$$\deg\psi(s)+d \leq \sum_{i=1}^{k_q}\epsilon_i+\sum_{i=1}^{k_\sigma}\sigma_i+\sum_{i=1}^{k_q}q_i+\sum_{i=1}^{k_p}p_i+\sum_{i=1}^{k_l}l_i, \tag{11}$$

$$\psi(s) \rhd \alpha_1(s)\alpha_2(s)\cdots\alpha_{k_l}(s), \tag{12}$$

$$d \geq \sum_{i=1}^{k_q}q_i+\sum_{i=1}^{k_p}p_i, \tag{13}$$

$$d = \sum_{i=1}^{k_p}p_i \tag{14}$$

with equality in (11) if $k_\epsilon = k_q$.

- The problem of pole assignment to the **column regularizable systems** was solved (under the assumption that $\alpha_i(s)$, the fixed invariant polynomials of the system, have the real roots.) The conditions of solvability are given in **Theorem 5**.

Theorem 5. [1, 2] *Let a column regularizable system (1) ($k_q \geq k_\epsilon$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$ be given. Then there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that, using the w-notation, $\tilde{\psi}(w)w^d = \mathrm{gcddm}\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$ if and only if the conditions (15)-(19) (and (20) if $k_\epsilon = 0$) are satisfied:*

$$\deg\psi(s) + d \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_\epsilon} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i, \tag{15}$$

$$\psi(s) \rhd \prod_{i=k_q-k_\epsilon+1}^{k_l} \alpha_i(s), \tag{16}$$

$$d \geq \sum_{i=1}^{k_\epsilon+k_p} z_i, \tag{17}$$

$$\deg\psi(s) \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_l} l_i, \tag{18}$$

$$d \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_\epsilon} q_i + \sum_{i=1}^{k_p} p_i, \tag{19}$$

$$d \leq \sum_{i=1}^{k_p} p_i, \tag{20}$$

with equality in (15) if $k_\epsilon = k_q$, and $\{z_i\}_{i=1}^{k_\epsilon+k_p}$ denotes the set of the first $k_\epsilon + k_p$ indices of the non-decreasingly ordered set $\{q_i\}_{i=1}^{k_q} \cup \{p_i\}_{i=1}^{k_p}$, and $\alpha_i(s) := 1$ for $k_l \leq k_q - k_\epsilon$.

- The necessary and sufficient conditions for the case when the maximal number of poles is to be assigned to a weakly regularizable system are given in **Theorem 6**.

- For a weakly regularizable system (1) it is possible to describe the poles by just the one dominant minor of $sE - A - BF$. The corresponding conditions are given in **Theorem 4** for the row regularizable systems and in **Proposition 5** for the column regularizable systems.

The extension of the regularizability to its weak analogue is due to the so-called *NS blocks*. In the case of row regularizable systems they generate $k_\epsilon - k_q$ nonproper indices. The column regularizable systems may possess $k_q - k_\epsilon$ NS q-blocks, and all the η-blocks belong to the NS blocks, too. Analyzing the results of pole assignment problem some remarks are given below.

- The value $\deg \psi(s) + d$ is the number of poles (multiplicities included) of system (1). In the case of regularizable system this value is constant (see (8)). The situation is different as far as weakly regularizable systems are concerned (see (11) and (15)).
- The maximal number of assignable poles of a weakly regularizable system,
$$\deg \psi(s) + d = \sum_{i=1}^{k_r} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i, \; k_r := \min\{k_\epsilon, k_q\},$$
cannot be increased by its NS blocks.
- The number of poles that can freely be assigned to the prescribed locations, say k_c, in the case of weakly regularizable systems can be different from the similar number in regularizable systems. Particularly, $k_c = \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i$ for the regularizable systems, while for the weakly regularizable systems $k_c \leq \sum_{i=1}^{k_r} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i, \; k_r := \min\{k_\epsilon, k_q\}$. It follows that NS ϵ-blocks or q-blocks may lead to the cancellation of all such poles (the case $k_c = 0$).
- The quantities $\alpha_i(s), q_i, p_i$ are unchanged by state feedback (2) in the case of regularizable and row regularizable systems. While in the column regularizable systems only $k_l - k_q + k_\epsilon$ smallest $\alpha_i(s)$ and $k_\epsilon + k_p$ smallest indices of $\{q_i\}_{i=1}^{k_q} \cup \{p_i\}_{i=1}^{k_p}$ are not changed.

5 The problem of pole structure assignment

A recent result of the problem of pole structure assignment for the square systems is stated by Loiseau and Zagalak in [22], where the authors give the necessary conditions to that problem which are also sufficient in some special cases.

Theorem 10. [22] *Given a regularizable system (1), monic polynomials* $\psi_1(s) \rhd \psi_2(s) \rhd \ldots \rhd \psi_r(s)$ *and positive integers* $d_1 \geq d_2 \geq \ldots \geq d_{k_d} > 0$*. Let* $\{h_i\}_{i=1}^{k_h}$ *stand for the non-increasingly ordered set of the positive indices* p_i *and* q_i *with* $k_h := \mathrm{card}\{h_i\}$ *and* $k := k_l + k_p + k_q$*. Let further* $\{c_i\}_{i=1}^{m}$ *denote the non-increasingly ordered set of the indices* ϵ_i *and* σ_i*,* $m := k_\epsilon + k_\sigma$*.*

Then, if there exists F *in (2) such that the matrix pencil* $sE - A - BF$ *is nonsingular and its finite and infinite zero structures are given by the polynomials* $\psi_i(s)$ *and the integers* d_i*, respectively, the following conditions hold:*

$$\psi_{i+m}(s) \lhd \alpha_i(s) \lhd \psi_i(s), \quad i = 1, 2, ..., k \tag{21}$$

$$d_{i+m} \leq h_i \leq d_i, \quad i = 1, 2, ..., k_h \tag{22}$$

$$\sum_{i=1}^{k+j} (\deg \beta_i^j(s) + \max(h_i, d_{i+m-j})) \leq \sum_{i=1}^{k} (\deg \alpha_i(s) + h_i) + \sum_{i=m-j+1}^{m} c_i, \; j = 1, 2, \ldots, m \tag{23}$$

$$k_d \leq k_\epsilon + \bar{k}_p \tag{24}$$

$$k_d \leq k_q + \bar{k}_p \tag{25}$$

and equality in (21) holds for $j = m$*, where*

$$\beta_i^j(s) := \mathrm{lcm}\,(\alpha_i(s), \psi_{i+m-j}(s)), \tag{26}$$

$\bar{k}_p := \mathrm{card}\{p_i > 0\}$*, and* $\alpha_i(s) := 1$ *for* $i > k_l$*,* $\psi_i(s) := 1$ *for* $i > k + m$*,* $h_i := 0$ *for* $i > k_h$*,* $d_i := 0$ *for* $i > k_d$*.*

Remark 1. The conditions (21) - (25) are also sufficient if:

- $\{p_i\} = \emptyset$ or $p_i = 0$ or $\{p_i\}$ is a subset of $\{d_i\}$, see [22];
- rank $E = n$, see [36];

- rank $[sE - A, -B] = q = n \ \ \forall s \in \mathbb{C} \cup \infty$, see [18];
- rank $E = n$ and rank $[sE - A, -B] = q = n \ \ \forall s \in \mathbb{C}$, see [31].

Pole structure assignment to the non-square systems

A possibility to assign the one nonzero dominant minor of $sE - A - BF$ to describe the poles of the closed loop system implies that the regular subpencil of $sE - A - BF$, which corresponds to that prescribed dominant minor, will define the pole structure of the system. Taking this into account, the result of the problem of pole structure assignment by state feedback known for the square systems was extended for the case of weakly regularizable systems (1) in the following way.

Theorem 11. *Given a weakly regularizable system (1), monic polynomials $\psi_1(s) \rhd \psi_2(s) \rhd \ldots \rhd \psi_r(s)$ and positive integers $d_1 \geq d_2 \geq \ldots \geq d_{k_d} > 0$ such that $\{p_i\}_{i=1}^{\bar{k}_p} \subset \{d_i\}_{i=1}^{k_d}$, $\bar{k}_p := \mathrm{card}\{p_i > 0\}$. Let $\{q_i'\}_{i=1}^{k_r}$ be a subset of $\{q_i\}_{i=1}^{k_q}$ and $\{\epsilon_i'\}_{i=1}^{k_r}$ be a subset of $\{\epsilon_i\}_{i=1}^{k_\epsilon}$, where $k_r := \min\{k_\epsilon, k_q\}$. Let further $\{h_i\}_{i=1}^{k_h}$ stand for the non-increasingly ordered set of the positive indices p_i and q_i', $k_h := \mathrm{card}\{h_i\}$, and $k := k_l + k_p + k_r$. Let further $\{c_i\}_{i=1}^{m}$ denote the non-increasingly ordered set of the indices ϵ_i' and σ_i, $m := k_r + k_\sigma$.*

Then, there exists F in (2) such that the finite and infinite zero structures of $sE - A - BF$ are given by the polynomials $\psi_i(s)$ and the integers d_i, respectively, if the following conditions hold:

$$\psi_{i+m}(s) \lhd \alpha_i(s) \lhd \psi_i(s), \quad i = 1, 2, ..., k \tag{27}$$

$$d_{i+m} \leq h_i \leq d_i, \quad i = 1, 2, ..., k_h \tag{28}$$

$$\sum_{i=1}^{k+j} (\deg \beta_i^j(s) + \max(h_i, d_{i+m-j})) \leq \sum_{i=1}^{k} (\deg \alpha_i(s) + h_i) + \sum_{i=m-j+1}^{m} c_i, \ j = 1, 2, \ldots, m \tag{29}$$

$$k_d \leq k_r + \bar{k}_p \tag{30}$$

and equality in (29) holds for $j = m$, where $\beta_i^j(s)$ are defined in (26), $\alpha_i(s) := 1$ for $i > k_l$, $\psi_i(s) := 1$ for $i > k + m$, $h_i := 0$ for $i > k_h$, $d_i := 0$ for $i > k_d$.

6 Controllability

The concept of controllability of the system (1) belongs to the most important ones in control theory. It describes our ability to change the behavior of the system by means of the control inputs. Therefore, it relates to the pole assignment problems by state feedback. Particularly, any controllable pole can be shifted to any prescribed location by choosing an appropriate state feedback gain. There are two main notions of controllability of the system (1). They are marked as R - and V- controllability in the text since they are also called as the controllability "in the sense of Rosenbrock" and "in the sense of Verghese", respectively.

Controllability is well studied for the case of square systems using different approaches (frequency-domain, time-domain, geometrical etc., see [6, 7, 19, 30, 31, 34, 35].

In the **time domain**, controllability is closely connected with the notion of reachability. Reachability of the system is the ability to get from one state to another state. Controllability is usually defined either of the following properties:

- *reachability of every state from any state* (R-controllability) [6, 35];
- *reachability of zero* (V-controllability) [19, 30].

R-controllability describes the ability to control the state trajectory of the whole system, while V-controllability do not take that of the non-dynamic subsystem (which associates to the blocks of infinite elementary divisors of order $\mu_i = 0$) into the consideration.

In the **frequency domain**, the concept of controllability is closely tied with the notion of irreducibility. But at first let us begin with the free-response modes (i.e. when $u(t) = 0$) of the system. In the square systems (1) these modes are given by the zeros of $sE - A$.

The modes of the system which can not be influenced by control inputs (decoupled from inputs) are called the *input decoupling zeros* [31] and associate to the (finite and infinite) zeros of $\begin{bmatrix} sE - A, & -B \end{bmatrix}$. If the pencil $\begin{bmatrix} sE - A, & -B \end{bmatrix}$ is of full row rank for all finite and infinite s then the system has no input decoupling zeros. This means that all its poles are controllable and the system is said to be strongly controllable [34] (V-controllable).

- A system (1) is irreducible if and only if

$$\operatorname{rank}\left[sE - A, \ \ -B\right] = q, \quad \forall s \in \mathbb{C} \cup \infty. \tag{31}$$

There is another way to treat the pole at $s = \infty$ (due to Rosenbrock) which leads to consider the modes of the non-dynamic subsystem as the infinite poles of (1). Then, irreducibility is equivalent to R-controllability.

That is to say, controllability of the regularizable systems coincide with

- *irreducibility in the sense of Rosenbrock* (R-controllability);
- *irreducibility in the sense of Verghese* (V-controllability).

Controllability of the non-square systems

Controllability of the non-square systems (1) was considered to be adapted to the task of problem of pole assignment by a proportional state feedback.

Time-domain approach

The non-square systems are no longer regularizable, hence they loose such an important property as the uniqueness of its solution. In particular, this occurs in the systems with $q < n$. The "measure of non-uniqueness" is gathered by the so called internal degree of freedom [3] of the system. This term represents variables of the state which are undetermined by the equation (1) and assumed to be described by some free functions. Then, which of the following properties of the system:

- the existence of the proper trajectory (which is not constrained by the equation (1)),
- the existence of the proper control (which is not guaranteed by the equation (1)),

is emphasized the results on reachability, and consequently, controllability follow. For example, the system can be considered to be reachable just due to its internal degree of freedom. This is the main divergence of the approaches to controllability for the non-square systems in [3, 14]. Therefore, the state uniqueness property, which characterizes the fact that state trajectories are uniquely fixed by the initial condition and input, was involved directly into the concepts of R- and V-controllability for the non-square systems. In particular,

- A system (1) is said to be *V-controllable* if for every state $x_T \in \mathbb{R}^n$ there exists an input $u(t)$ and a final time T such that there exists a unique state trajectory $x(t)$ on $[0, T]$ with $x(0) = x_T$ and $x(T) = 0$.

Using the introduced definitions, the conditions for the system to be R-controllable and V-controllable were established in **Theorems 17, 18**, respectively.

Theorem 18. [3] *A system (1) is V-controllable if and only if (the following condition are equivalent):*

a).

$$\begin{aligned} &(k_q \geq k_\epsilon) \wedge (k_l = 0) \wedge \big((k_p = 0) \dot{\vee} (p_i = 0)\big) \wedge \\ &\big((k_q = 0) \dot{\vee} (q_i = 0)\big) \wedge \big((k_\eta = 0) \dot{\vee} (\eta_i = 0)\big) \end{aligned} \tag{32}$$

b).

$$\bar{q} \geq n, \tag{33}$$

$$\text{Im}\bar{E} + \text{Im}\bar{B} + \bar{A}\text{Ker}\bar{E} = \mathbb{R}^{\bar{q}}, \tag{34}$$

$$\text{Im}[s\bar{E} - \bar{A}] + \text{Im}\bar{B} = \mathbb{R}^{\bar{q}} \quad \forall s \in \mathbb{C}, \tag{35}$$

where $[\bar{E}\ \bar{A}\ \bar{B}] \in \mathbb{R}^{\bar{q}\times n}$ *is of full row rank matrix such that*

$$[E \quad A \quad B] = \begin{bmatrix} I_{\bar{q}} \\ Y \end{bmatrix} [\bar{E} \quad \bar{A}, \quad \bar{B}], \tag{36}$$

with $\bar{E}, \bar{A} \in \mathbb{R}^{\bar{q}\times n}$, $\bar{B} \in \mathbb{R}^{\bar{q}\times m}$, $Y \in \mathbb{R}^{(q-\bar{q})\times\bar{q}}$.

The conditions (33)-(35) can be written in a more condensed form,

$$\bar{q} \geq n, \quad \text{rank}\left[s\bar{E} - \bar{A} \quad \bar{B}\right] = \bar{q} \quad \forall s \in \mathbb{C} \cup \infty. \tag{37}$$

Frequency-domain approach

In the non-square systems the free-response modes of the system are not given just by the zeros of $sE - A$. In particular, using the introduced above terminology, the modes corresponding to the NS blocks of $sE - A$ don't associate to the poles of the system.

By (37), *the concepts of irreducibility (in the sense of Verghese) and V-controllability are still equivalent for the systems satisfying* $\bar{q} \geq n$*.* In other case these concepts don't coincide. Let us consider this point in more details.

By Theorem 18, a column regularizable system ($k_q \geq k_\epsilon$) is not V-controllable iff $(k_l \neq 0) \vee (p_i > 0) \vee (q_i > 0) \vee (\eta_i > 0)$ for some i ($\vee$ means OR). Its NS blocks ($k_q - k_\epsilon$ of the indices $q_i > 0$ and the indices $\eta_i > 0$) admit only the zero solution. Hence, the free response modes given by these indices can be considered to be called the *constrained modes*. Their presence implies non-irredicibility of the system (the non-coprimeness of $sE - A$ and B). Therefore, for the column regularizable systems the meaning of irreducibility is the same as for the square systems - the absence of the input decoupling zeros or equivalently the absence of the uncontrollable modes.

The presence of internal degree of freedom in the systems with $q < n$ (in the row regularizable systems) implies the existence of free response modes which are not fixed by the initial condition and decoupled from the input. They were called the *non-fixed modes*, and correspond to $k_\epsilon - k_q$ non-proper indices.

Then, the following definition of the V-controllable system in the frequency domain is consistent with that in the time domain:

- The system (1) is *V-controllable (strongly controllable)* if it is irreducible and has no non-fixed modes.

The state uniqueness property, which was involved directly into the the concept of controllability for the rectangular systems (1) in the time-domain, sounds here as the absence of the non-fixed modes.

Controllability indices

The concept of controllability of the system (1) can also be considered using the notion of reachability and controllability indices. The space of all states of the system which are reachable from zero is called the reachable space. Let it be denoted by $\mathcal{R}$. It is shown in [25] that for the square systems the reachability indices are dimensions of singly generated reachability subspaces into which the reachable space of the system $\mathcal{R}$ can be split, while the controllability indices are related to a splitting of $E\mathcal{R}$.

Let ε_i denote the column minimal indices of the pencil $[sE - A, \ -B]$ and ϕ_i the column minimal indices of the pencil $\Pi[sE - A]$, where Π is a maximal left annihilator of B. Then, the geometric characterization of the indices mentioned above corresponds to the following polynomial characterization [25]:

- the reachability indices, $\{r_i\}$ are given (modulo ordering) by $\{r_i\} = \{1 + \phi_i\}$;
- the controllability indices, $\{c_i\}$ are given (modulo ordering) by $\{c_i\} = \{\varepsilon_i \mid \varepsilon_i \neq 0\}$.

Theorem 19. [25] *A regularizable system (1) is V-controllable (R-controllable) iff the condition a) (the condition b), respectively) holds:*

a) $\sum_i c_i = n - \dim \mathrm{Ker} E$;

b) $\sum_i r_i = n$.

Controllability indices for the non-square systems

Concerning the non-square systems, the above characterization of the indice doesn't always bring the same information as in the case of square systems. In particular, the lack of correspondence appears in the systems with internal degree of freedom. The reachability and controllability subspaces of the system which are generated by its internal degree of freedom start in $\mathrm{Ker}E$ and not in $E^{-1}\mathrm{Im}B/\mathrm{Ker}E$ in opposition with the other subspaces which are directly generated by some input [3].

In fact, it can be easily noted that the condition a) of Theorem 19 implies irreducibility of (E, A, B) (in the sence of Verghese), while the condition b) - irreducibility in the sense of Rosenbrock.

Hence, the extension of the notion of controllability and reachability indices for the rectangular systems conformably with the problem of pole assignment can be done just in the case of the column regularizabe systems.

7 Conclusions

Thesis deals with rectangular descriptor systems of the form (1). In particular, the effect of a proportional state feedback (2) upon the system was investigated. The main problems under consideration are the problems of pole and pole structure assignment for the non-square systems (1) by state feedback. These problems are of significant value in control theory since they aim at shaping the desired closed-loop system response.

In the rectangular systems, the concept of weak regularizability-the assumption on the existence of a state feedback gain F such that the pencil $sE - A - BF$ is of full row or column rank - was introduced. Accordingly, the rectangular systems are subdivided into the row and column regularizable systems, see Theorem 2. Weak regularizability is a reasonable property of (1) and is the extension of reguarizability known for the square systems.

Then, the following results were reached for the problem of pole assignment in the case of weakly regularizable systems (1):

- $k_\eta = 0$ and $k_\epsilon \geq k_q$ - Theorem 3 (necessary conditions) and Theorem 4 (sufficient conditions for the particular cases)
- $k_q \geq k_\epsilon$ - Theorem 5 (necessary and sufficient (under assumptions of reals roots of the fixed invariant polynomials) conditions)

The result of pole structure assignment known for the square systems was extended for the rectangular systems in the special cases. In particular, the cases when the specific subpencil of $sE - A - BF$ is to be assigned to describe the pole structure of the closed loop system have been considered. As a result, the sufficient conditions for that cases were established for the systems satisfying

- $k_\eta = 0$ and $k_\epsilon \geq k_q$ or $k_q \geq k_\epsilon$ - Theorem 11 (sufficient conditions for the special cases)

Two notions of controllability of the system (1) occurring in the literature, which were termed as R - and V - controllability, were studied through the time and frequency domain approaches. These concepts were adapted to the problem of pole assignment to the system (1) by a proportional state feedback (2). The corresponding conditions are given in terms of feedback invariants as well as in geometric terms, see Theorem 17 (R - controllability) and Theorem 18 (V - controllability). In fact, the properties describing the square systems to be controllable through the frequency-domain such as the absence of input decoupling zeros, irreducibility, are not well fitted for the general case of (1). This is caused by that rectangular systems, in general, posses some special features like the existence of the so called non-fixed and constrained modes.

The notions of controllability indices are conformable extended just for the column regularizable systems.

7.1 Suggestions for future research

Some open questions in the problems of pole and pole structure assignment to the rectangular systems (1) by the proportional state feedback (2) still exist and could be considered for investigation in the future. Namely:

- poles assignment problem - give sufficient conditions in the case of row regularizable systems ($k_\eta = 0$ and $k_\epsilon \geq k_q$);
- pole structure assignment problem - there is no complete solution in the case of column regularizable systems ($k_q \geq k_\epsilon$);
- the results of [9] are somewhat implicit and obtained using another approach as that of [22] (which was used here) and stated in different notations - their restatement can be useful for the application in control.

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